

Sr. No. 1131

C-HLR-K-TA

## STATISTICS – I

Time Allowed : Three Hours

Maximum Marks : 200

## INSTRUCTIONS

*Candidates should attempt FIVE questions in ALL including Question Nos. 1 and 5 which are compulsory. The remaining THREE questions should be answered by choosing at least ONE question each from Section A and Section B.*

*The number of marks carried by each question is indicated against each.*

*Answers must be written only in ENGLISH.*

*(Symbols and abbreviations are as usual)*

*Any essential data assumed by candidates for answering questions must be clearly stated.*

## Section – A

1. Answer any *five* of the following :  $8 \times 5 = 40$

- (a) An unbiased die is rolled twice. Let  $A$  be the event that the first throw shows a number  $\leq 2$ , and  $B$  be the event that the second throw shows at least 5.

Show that  $P(A \cup B) = \frac{5}{9}$ .

(Contd.)

- (b) Let  $\{A_n\}$  be a non-decreasing sequence of events. Show that :

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\lim_{n \rightarrow \infty} A_n\right).$$

- (c) Let  $X$  be a random variable defined on  $(\Omega, A, P)$ . Define a point function  $F(x) = P\{\omega : X(\omega) \leq x\}$ , for all  $x \in R$ . Show that the function  $F$  is indeed a distribution function.

- (d) Let  $k > 0$  be a constant, and

$$f(x) = \begin{cases} kx(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

obtain  $P(X > 0.3)$ .

- (e) Let  $X$  be a random variable with  $E(X) = 0$  and  $\text{Var}(X) = \sigma^2$ .

Show that :

$$P(X \geq x) \leq \frac{\sigma^2}{\sigma^2 + x^2} \text{ if } x > 0$$

$$\text{and } P(X \geq x) \geq \frac{x^2}{\sigma^2 + x^2} \text{ if } x < 0.$$

- (f) Let  $X_1, X_2, \dots$ , be independent and identically distributed random variables with common p.d.f.

$$f(x) = \begin{cases} \frac{1+\delta}{x^{2+\delta}}, & x \geq 1, \delta > 0. \\ 0, & x < 1 \end{cases}$$

Show that law of large numbers holds.

- (a) Let  $\{X_n, Y_n\}$ ,  $n = 1, 2, \dots$  be a sequence of random variables. Then

$$|X_n - Y_n| \xrightarrow{P} 0 \text{ and } Y_n \xrightarrow{\alpha} Y \Rightarrow X_n \xrightarrow{\alpha} Y.$$

Prove it.

- (b) Let  $\beta_n = E|X|^n < \infty$ . Show that for arbitrary  $k$ ,  $2 \leq k \leq n$ ,

$$\beta_{k-1}^{\frac{1}{k-1}} \leq \beta_k^{\frac{1}{k}}$$

- (c) Let  $(X, Y)$  be jointly distributed with p.d.f.

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find marginal probability density functions of  $X$  and  $Y$ .

- (d) Let  $X_1, X_2, \dots, X_m$  be i.i.d. random variables with common p.m.f.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots, n, \\ 0 < p < 1.$$

Obtain the p.m.f. of  $S_m = X_1 + X_2 + \dots + X_m$ .

10×4=40

3. (a) Let  $X$  have a geometric distribution, then for any two non-negative integers  $m$  and  $n$ ,

$$P(X > m+n / X > m) = P(X \geq n).$$

Prove it.

- (b) Let  $X$  be a random variable with a continuous distribution function  $F$ . Show that  $F(X)$  has the uniform distribution on  $(0, 1)$ .

- (c) Let  $X$  be a Poisson random variable with parameter  $\lambda$ . Show that

$$P(X \leq K) = \frac{1}{K!} \int_{\lambda}^{\infty} e^{-x} x^K dx.$$

- (d) How large a sample must be taken in order that the probability will be at least 0.95 that  $\bar{X}_n$  will be within 0.5 of  $\mu$ . ( $\mu$  is unknown and  $\sigma = 1$ ). 10×4=40

4. (a) Let  $F_n(x)$  be the distribution function defined by

$$F_n(x) = \begin{cases} 0, & \text{for } x \leq -n \\ \frac{x+n}{2n}, & \text{for } -n < x < n \\ 1, & \text{for } x \geq n. \end{cases}$$

Is the  $\lim_{n \rightarrow \infty} F_n(x)$  a distribution function? If not, why?

- (b) If  $X_i$  can have only two values with equal probabilities  $i^\alpha$  and  $-i^\alpha$ , show that law of large numbers can be applied to the independent variables  $X_1, X_2, \dots$ , if  $\alpha < \frac{1}{2}$ .
- (c) If  $A_1, A_2, \dots$ , be a sequence of events on the probability space  $(S, B, P)$  and let  $A = \overline{\lim} A_n$ .

If  $\sum_{n=1}^{\infty} P(A_n) < \infty$ , then  $P(A) = 0$ .

Prove it.

- (d) Let  $\{X_k\}$  be independent random variables with p.m.f.

$$P(X_k = k^\lambda) = P(X_k = -k^\lambda) = \frac{1}{2}, \quad k = 1, 2, \dots$$

Show that this sequence  $\{X_k\}$  obeys central limit theorem, for  $\lambda > 0$ . 10×4=40

### Section – B

5. Answer any *five* of the following : 8×5=40

(a) Show that for any discrete distribution, standard deviation is not less than mean deviation from mean.

(b) Let  $(X, Y)$  be jointly distributed with density function

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1; \quad 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Obtain correlation coefficient between  $X$  and  $Y$ .

(c) For the  $2 \times 2$  table

$a$	$b$
$c$	$d$

prove that Chi-square test of independence gives

$$\chi^2 = \frac{N(ad - bc)}{(a + c)(b + d)(a + b)(c + d)},$$

$$N = a + b + c + d.$$

- (d) The following data represent lifetimes (hours) of batteries for two different brands :

Brand A :	40	30	40	45	55	30
Brand B :	50	50	45	55	60	40

Test whether two brands are the same.

- (e) Estimate  $U_2$  from the following table :

$x$ :	1	2	3	4	5
$U_x$ :	7	-	13	21	37

- (f) Evaluate  $\log_e 7$  by Simpson's  $\frac{1}{3}$ rd rule.

6. (a) A cyclist pedals from his house to his college at a speed of 10 km per hour and back from the college to his house at 15 km per hour. Find the average speed.

- (b) The random variables  $X$  and  $Y$  are jointly normally distributed and  $U$  and  $V$  are defined by  $U = X \cos \alpha + Y \sin \alpha$ ,  $V = Y \cos \alpha - X \sin \alpha$ . Show that  $U$  and  $V$  will be uncorrelated if

$$\tan 2\alpha = \frac{2\gamma\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

- (c) The following values of the function  $f(x)$  for values of  $x$  are given :

$$f(1) = 4, f(2) = 5, f(7) = 5, f(8) = 4.$$

Find the value of  $f(6)$  and also the value of  $x$  for which  $f(x)$  is maximum or minimum.

- (d) If third differences are constant, prove that

$$\int_{-1}^1 f(x) dx = \frac{2}{3} \left[ f(0) + f\left(\frac{1}{\sqrt{2}}\right) + f\left(-\frac{1}{\sqrt{2}}\right) \right]$$

$$10 \times 4 = 40$$

7. (a) The first three moments of a distribution about the value 1 are 2, 25 and 80. Find its mean, standard deviation and the moment-measure of skewness.
- (b) You are working as a purchase manager for a company. The following information has been supplied to you by two manufacturers of electric bulbs.

	<i>Company A</i>	<i>Company B</i>
Mean life (in hours)	1300	1288
Standard deviation (in hours)	82	93
Sample size	100	100

Which brand of bulbs are you going to purchase if you desire to take a risk of 5% ?

- (c) Describe clearly sign test. State its asymptotic relative efficiency with respect to  $t$ -test.
- (d) The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable ? 10×4=40

8. (a) Show that for discrete distribution  $\beta_2 > 1$ .
- (b) Find the most likely price in Mumbai corresponding to the price of Rs. 70 at Kolkata from the following :

	<i>Kolkata</i>	<i>Mumbai</i>
Average price	65	67
Standard deviation	2.5	3.5

Correlation coefficient between the prices of commodities in the two cities is 0.8.

- (c) Show that for  $t$ -distribution with  $n$  degrees of freedom, mean deviation about mean is given by

$$\frac{\sqrt{n} \Gamma\left[\left(\frac{n-1}{2}\right)\right]}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right)}$$

- (d) If  $f(x) = \frac{1}{x^2}$ , find the divided differences  $f(a, b)$  and  $f(a, b, c)$ . 10×4=40